

All-Order Theory of Continuous Quantum Noise Measurements: Application to Spin Noise Spectroscopy

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The stochastic detector output $z(t)$ of a general continuously measured quantum system is calculated using a stochastic master equation approach. The resulting continuous quantum noise formula (CQNF) is valid in all orders of the measurement strength. The CQNF provides a rigorous route to the calculation of higher order noise spectra of quantum systems with applications in spin noise spectroscopy and quantum measurement theory in general.

Within classical physics measurements on a physical system can in principle be performed with arbitrary precision and vanishing perturbation of the system. In contrast, within quantum mechanics both stochastic measurement outcomes for identical initial conditions and back-action of the measurement on the quantum system are identified as fundamental properties of nature. After a single measurement the wave-function collapses into a quantum state that corresponds to the measurement result. Apart from such a strong measurement, so-called weak continuous measurements of an operator A can be realized where a quantum system is continuously monitored with partial collapses taking place during finite time intervals yielding a time-dependent detector output $z(t)$. Expressions for $z(t)$ and its statistical properties have in the past been provided only for special systems and in limited orders of the measurement strength (e.g. [1–5]). We provide here expressions for the stochastic detector output $z(t)$ for general quantum systems in all orders of the measurement strength. We will choose the celebrated method of spin noise spectroscopy as an illustrative example along which we will identify important quantities of the theory in the experimental setup. Spin noise spectroscopy has after pioneering work by Aleksandrov and Crooker [6, 7] quickly evolved into a mighty tool for studying spin systems in semiconductors and gases [8–12]. The spin noise setup shown in Fig. 1 realizes a continuous quantum measurement of the z -spin orientation of an electron in a semiconductor sample. The electron may in general be part of a larger coupled quantum system like e.g. an interacting pair of an electron spin and a nuclear spin [13]. A finite Faraday-angle of the probe light polarization after the sample leads to a slight imbalance of light intensities on the photodiodes. The Faraday signal is superimposed by strong optical shot noise. The z -spin orientation in the quantum system is consequently not instantly revealed by the Faraday effect as expected for a weak quantum measurement. The (spin) noise power spectrum of $z(t)$ is usually defined via

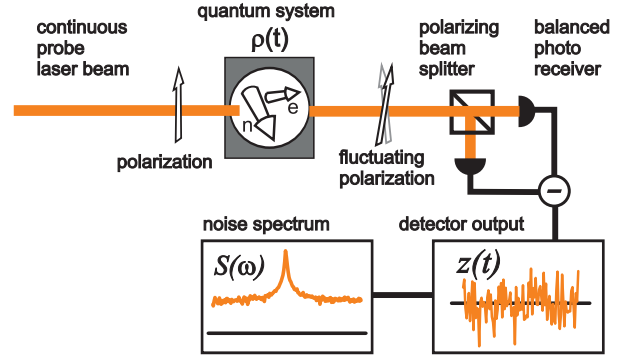


FIG. 1. Schematics of continuous spin noise spectroscopy. The fluctuating electron spin orientation in a quantum system induces a fluctuating polarization of the probe beam. Polarization noise and photon shot noise contribute to the noise signal $z(t)$.

an autocorrelation as

$$S(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega\tau}}{2\pi} (\langle z(t+\tau)z(t) \rangle - \langle z(t+\tau) \rangle \langle z(t) \rangle) d\tau \quad (1)$$

or equivalently via [14]

$$\langle z(\omega)\overline{z(\omega')} \rangle - \langle z(\omega) \rangle \langle \overline{z(\omega')} \rangle = \delta(\omega - \omega') S(\omega) \quad (2)$$

where $z(\omega)$ is the Fourier transform of $z(t)$ and $\overline{z(\omega')}$ denotes the complex conjugate of $z(\omega')$. Spin precession frequencies and spin lifetimes of electron spins can be deduced from the peak position and peak width of the power spectrum. Recently, efforts were made to explore the possibility of higher order noise spectra for obtaining additional information on quantum systems. Liu showed theoretically that third order moments of the noise signal can be used to distinguish homogenous from inhomogeneous broadening of the spin noise resonance [15]. First steps into developing a theoretical understanding of a fourth order spin noise spectrum

$$S(\omega_1, \omega_2) = \langle |z(\omega_1)|^2 |z(\omega_2)|^2 \rangle - \langle |z(\omega_1)|^2 \rangle \langle |z(\omega_2)|^2 \rangle \quad (3)$$

were undertaken by Li and Sinitsyn [16, 17]. Starosielec *et al.* presented a first radio frequency setup for practical broadband real-time measurements of $S(\omega_1, \omega_2)$ [18].

Despite the growing popularity of spin noise spectroscopy, theoretical approaches are mainly limited to semiclassical descriptions of the spin dynamics that provide only a phenomenological treatment of spin relaxation and neglect the back-action of the quantum measurement on the system [19]. No quantitative statements were possible regarding e.g. the ratio of the shot noise background signal and the desired spin noise signal. Our general theory covers here for the first time the problem fully quantum mechanically providing a rigorous route for the calculation of any higher order noise spectrum for a given quantum system.

In 1987, Belavkin published a set of stochastic differential equations that treat both the detector output and the evolution of the quantum system taking into account the effect of a continuous measurement [20]. Here, we use his theory in the form of a stochastic master equation (SME) in the Schrödinger picture (Itô-calculus)

$$\begin{aligned} d\rho = & \frac{i}{\hbar}[\rho, H]dt + \mathcal{D}\rho dt \\ & - \frac{\beta^2}{2}[A, [A, \rho]]dt \\ & + \lambda\beta[A\rho + \rho A - 2\rho\text{Tr}(\rho A)]dW \end{aligned} \quad (4)$$

that propagates the density matrix $\rho(t)$ of a quantum system which is continuously monitored for the expectation value of an operator A [21]. Corresponding equations were also derived by Korotkov and Goan for special cases in transport theory and quantum optics, respectively [1, 3]. A second stochastic equation

$$dZ = \beta^2\text{Tr}(\rho A)dt + \beta\frac{1}{2}dW \quad (5)$$

describes the evolution of the time-integrated detector output $Z(t)$. The first line of the SME corresponds to a usual master equation with Hamilton operator H and a linear superoperator \mathcal{D} that models damping of the quantum system due to coupling with its environment. The continuous measurement of operator A leads to a decay of the system towards an eigenstate of A . The gradual decay appears as an additional damping term in the second line of the SME (see also [5]). The strength of the damping scales with β^2 , where β is in the case of optical spin noise spectroscopy proportional to the laser field amplitude and its coupling strength to the electron spin. Korotkov recovered in a numerical example of a two level system a telegraph noise behavior of the detector output for large coupling as expected for strong quantum measurements [2]. The quantity dW is a zero-mean Gaussian random variable with variance dt . It is this quantity through which a correct description of the randomness in quantum measurements appears in the theory. The detector output [Eq. (5)] consists of the expectation value of A and Gaussian noise dW . The latter contribution corresponds in the case of spin noise spectroscopy to photon shot noise and is present even with no sample in the beam

path (i.e. $A = 0$). The last line of the SME establishes the back-action of the stochastic measurement result on ρ . Please note that the last line is non-linear in ρ . The appearance of such a non-linearity is no surprise as any quantum measurement destroys quantum linearity [22]. The parameter λ will below be used for a successive approximation of the solution for $\rho(t)$. The above equations are our starting point for calculating the fluctuating detector output in all orders of β and eventually for finding expressions for noise spectra of deliberate quantum systems.

In a first step we will - without any approximation - eliminate the non-linear term in the SME and then derive an analytical expression for the time dependent detector output $z(t) = \dot{Z}(t)$ for all orders of β . The temporal derivative $\Gamma(t) = \dot{W}$ can be formally introduced and corresponds to white Gaussian noise (Langevin noise) with the property $\langle \Gamma(t)\Gamma(t') \rangle = \delta(t - t')$. The first two lines of the SME will be abbreviated with $\dot{\rho} = \mathcal{L}\rho$ where \mathcal{L} is a linear superoperator (the Liouvillian).

The nonlinear term in the SME

$$\dot{\rho} = \mathcal{L}(\rho) + \lambda\beta(A\rho + \rho A - 2\rho\text{Tr}(\rho A))\Gamma(t) \quad (6)$$

is eliminated after introducing an unnormalized density matrix u with

$$\rho(t) = u(t)e^{f(t)}. \quad (7)$$

that leads to a new linear stochastic master equation

$$\dot{u} = \mathcal{L}(u) + \lambda\beta(Au + uA)\Gamma(t) \quad (8)$$

and a second equation

$$\dot{f} = -2\lambda\beta\text{Tr}(ue^f A)\Gamma(t). \quad (9)$$

for $f(t)$ (a similar procedure is found in Ref. [21]). Knowledge of f is not needed in the following as the quantity of interest $\rho(t)$ can be obtained from $u(t)$ via

$$\rho(t) = \frac{u(t)}{\text{Tr}(u(t))} \quad (10)$$

simply by the requirement $\text{Tr}(\rho(t)) = 1$.

Next, we solve Eq. (8) by the method of successive approximation using the ansatz

$$u = u_0 + \lambda u_1 + \lambda^2 u_2 + \dots \quad (11)$$

which gives

$$\dot{u}_{n+1} - \mathcal{L}(u_{n+1}) = \beta(Au_n + u_n A)\Gamma(t). \quad (12)$$

The zero order contribution $u_0(t)$ fulfills

$$\dot{u}_0 = \mathcal{L}(u_0) \quad (13)$$

and reaches a constant equilibrium u_0 for $t \rightarrow \infty$ due to damping. We can therefore consider $\rho_0 = u_0$ (with

$\text{Tr}(u_0) = 1$) as the thermal equilibrium state of the continuously monitored quantum system.

Eqs. (12) can be interpreted as first order linear differential equations for u_{n+1} with driving terms that depend on u_n . Consequently, eqs. (12) can be solved after introducing a superoperator $\mathcal{G}(t) = \exp(\mathcal{L}t)$ by

$$u_n = \beta \mathcal{G}(t)^* (\Gamma(t)(Au_{n-1}(t) + u_{n-1}(t)A)) \quad (14)$$

where $*$ denotes a convolution in time. This simplifies further if a superoperator $\mathcal{J}(t) = \mathcal{G}(t)\mathcal{A}$ with $\mathcal{A}u = Au + uA$ is introduced to

$$u_n = \beta \mathcal{J}(t)^* (\Gamma(t)u_{n-1}(t)). \quad (15)$$

The detector output

$$z(t) = \beta^2 \text{Tr}(A\rho(t)) + \frac{1}{2}\beta\Gamma(t) \quad (16)$$

with

$$\rho(t) = \frac{\sum_{\nu=0}^{\infty} \beta^\nu \overbrace{\mathcal{J}(t)^* (\Gamma(t) (\cdots))}^{\nu\text{-times}} \rho_0}{\text{Tr}(\sum_{\nu=0}^{\infty} \beta^\nu \mathcal{J}(t)^* (\Gamma(t) (\cdots)) \rho_0)}$$

follows - as the central result of this work - for *all* orders of β . We will refer to Eq. (16) as the continuous quantum noise formula (CQNF). The CQNF serves as starting point for calculating moments, cumulants, and noise spectra of the detector signal $z(t)$ in any desired order of β . Please note that the denominator in the equation for $\rho(t)$ causes a non-linear dependence on ρ_0 as expected for a quantum system subject to measurements.

Before calculating noise spectra, we emphasize that fluctuating quantities like $z(t)$ need to be characterized via so-called cumulants [14]

$$\begin{aligned} C_1(X_i) &= \langle X_i \rangle \\ C_2(X_i, X_j) &= \langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle \\ C_3(X_i, X_j, X_k) &= \langle X_i X_j X_k \rangle - \langle X_i X_j \rangle \langle X_k \rangle \\ &\quad - \langle X_i X_k \rangle \langle X_j \rangle - \langle X_j X_k \rangle \langle X_i \rangle + 2\langle X_i \rangle \langle X_j \rangle \langle X_k \rangle \end{aligned}$$

which are special products of moments of stochastic quantities X_i . Any cumulant shares the important property that $C(X_i + Y_i, \dots) = C(X_i, \dots) + C(Y_i, \dots)$ provided that X_i and Y_i are *independent* stochastic quantities. Any noise spectrum of $z(t)$ is therefore required to be a cumulant of $z(t)$ in order to guarantee that e.g. the spectrum of a spurious background noise $z_b(t)$ can be subtracted from the spectrum of the measured quantity $z(t) + z_b(t)$. Any ordinary moment of the measured quantity would suffer from spurious cross-contributions of $z(t)$ and $z_b(t)$.

We obtain the second order noise spectrum after considering the terms of the Taylor expansion $z(t) = z_1(t) + z_2(t) + z_3(t) + O(\beta^4)$ in orders of β

$$\begin{aligned} z_1(t) &= \frac{1}{2}\beta\Gamma(t) \\ z_2(t) &= \beta^2 \text{Tr}(A\rho_0) \\ z_3(t) &= \beta^3 \text{Tr}[(A - \text{Tr}(A\rho_0))\mathcal{J}(t)^*(\Gamma(t)\rho_0)]. \quad (17) \end{aligned}$$

In the case of a spin noise experiment (i.e. $A = \sigma_z$), z_1 is due to laser shot noise, z_2 is a constant Faraday rotation due to a non-vanishing average spin-orientation in thermal equilibrium, and z_3 describes noise from the spin system. After Fourier transformation

$$\begin{aligned} z_1(\omega) &= \frac{1}{2}\beta\Gamma(\omega) \\ z_2(\omega) &= \beta^2 \text{Tr}(A\rho_0)\delta(\omega) \\ z_3(\omega) &= \beta^3 \text{Tr}[(A - \text{Tr}(A\rho_0))\mathcal{J}(\omega)\Gamma(\omega)\rho_0] \quad (18) \end{aligned}$$

the detector noise spectrum $S(\omega)$ follows from the cumulant

$$C_2(z(\omega), \overline{z(\omega')}) = \underbrace{\frac{\beta^2 S_{\text{sn}} + \beta^4 S_{\text{q}}(\omega)}{2\pi}}_{S(\omega)} \delta(\omega - \omega') + O(\beta^5) \quad (19)$$

which yields the desired second order noise spectrum

$$S_{\text{q}}(\omega) = \frac{1}{2} (\text{Tr}[(A - \text{Tr}(A\rho_0))\mathcal{J}(\omega)\rho_0] + \text{c.c.}) \quad (20)$$

of the observable A and a system independent shot noise contribution

$$S_{\text{sn}} = \frac{1}{4} \quad (21)$$

with flat spectrum (white noise). While S_{sn} followed from laser shot noise z_1 , the noise spectrum S_{q} originates from the correlation of shot noise z_1 and system noise z_3 . Corresponding relations for the special case of a two level system were previously found by Korotkov [2]. Our theory correctly predicts that the contribution of S_{sn} to $S(\omega)$ scales with the laser intensity β^2 , while S_{q} grows quadratically with the laser intensity. Another useful representation of S_{q} in terms of the Fourier transform of an autocorrelation-like expression $G(\tau)$ can be obtained from Eq. (20) as (see supplement [23])

$$S_{\text{q}}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\tau) e^{-i\omega\tau} d\tau \quad (22)$$

where

$$G(\tau) = \frac{1}{2} \langle A(|\tau|)A(0) + A(0)A(|\tau|) \rangle - \langle A(|\tau|) \rangle \langle A(0) \rangle. \quad (23)$$

The equation of motion for A in the Heisenberg picture can according to Lindblad always be obtained via a transformation of the original master equation for ρ [see Eq. (13)] as long as it is of the very general Lindblad type [24]. Consequently, the treatment of damping is possible in both the Schrödinger and the Heisenberg picture.

While $G(\tau)$ was here derived rigorously from a continuous measurement theory, different versions of $G(\tau)$ have been used in literature as starting point for calculating noise spectra. The recent review by Clerk *et al.* defines a *symmetrized quantum noise spectral density* [4] where

$G^{(C)}(\tau) = \frac{1}{2}\langle A(\tau)A(0) + A(0)A(\tau) \rangle$ assuming $\langle A(\tau) \rangle = 0$. $G^{(C)}(\tau)$ and $G(\tau)$ coincide only in the case of time symmetry due to the missing modulus function applied to τ . Any damping mechanism acting on the quantum system, however, breaks time symmetry leading to different spectra calculated from $G^{(C)}(\tau)$ and $G(\tau)$. Braun and König calculate spin noise spectra with an expression equivalent to ours [25].

Quantum mechanical expression for polyspectra which are the most general form of higher order spectra require the calculation of cumulants $C_n(z(\omega_1), z(\omega_2), \dots, z(\omega_n))$ [26]. Their calculation from $z(t)$ in orders of β faces no fundamental obstacle except for the vast number of terms that need to be treated. Compact expressions like Eq. (20) may soon be found for the cases $n = 3$ and $n = 4$ with the help of computer algebra. We emphasize that the lowest order contribution to C_3 requires already the treatment of $z(\omega)$ in fourth order of β : All cumulants C_n with $n > 2$ vanish for $z_1(\omega)$, $z_2(\omega)$, and $z_3(\omega)$ as they are either Gaussian or constant [14]. The term z_4 , however, is quadratic in Γ yielding the lowest order non-Gaussian contribution to $z(\omega)$.

Last, we discuss an earlier result on time correlation functions in continuous quantum measurement. In 2012, Bednorz *et al.* derived expressions for higher order moments of the measurement result $a(t)$ (corresponding to $z(t)$ here) and found

$$\langle a(t_1) \cdots a(t_n) \rangle_q = \text{Tr}[\mathcal{A}U(t_n, t_{n-1}) \cdots \mathcal{A}U(t_2, t_1) \mathcal{A}U(t_1, 0) \rho_0], \quad (24)$$

where their $\mathcal{A}u = (Au + uA)/2$ and U is a propagator for the system ρ [5, 12]. Their theory is based on path integral like expressions for the probability of finding a specific time series $a(t)$. An explicit stochastic expression for $a(t)$ like the CQNF for $z(t)$ was not required for obtaining Eq. (24). Obviously, all higher moments of $a(t)$ depend linearly on ρ_0 , while in our theory $z(t)$ and therefore also all higher moments of $z(t)$ depend in general non-linearly on ρ_0 [e.g. $\langle z_1(t)z_3(t') \rangle$ is quadratic in ρ_0]. It is at present unclear to us if this possible discrepancy of the theories can be reconciled or not.

In conclusion, we derived the continuous quantum noise formula as a general solution to the continuous measurement problem. An expression for the second order quantum mechanical noise power spectrum S_q was derived in the Schrödinger picture. Another unambiguous expression for $S_q(\omega)$ that is similar (but not always identical) to existing expressions was found after transformation into the Heisenberg picture. The CQNF paves the way for systematically deriving expressions of higher order spectra of any desired order. Future work will regard the search for compact expressions of higher order spectra, numerical treatment of model systems, and the relation of the CQNF approach to earlier approaches.

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SUPPLEMENT: EXPRESSION FOR S_q IN THE HEISENBERG PICTURE

Consider the Fourier transform of S_q [see Eq. (20)]:

$$G(\tau) = \int_{-\infty}^{\infty} S_q(\omega) e^{i\omega\tau} d\omega \quad (25)$$

Writing $\mathcal{J}(t)X = \mathcal{G}(t)(AX + XA)$ we find

$$\begin{aligned} 2G(\tau) &= \text{Tr}(A\mathcal{G}(\tau)(A\rho_0 + \rho_0 A)) \\ &\quad - \text{Tr}(A\rho_0)\text{Tr}(\mathcal{G}(\tau)(A\rho_0 + \rho_0 A)) \\ &\quad + \text{Tr}(A\mathcal{G}(-\tau)(A\rho_0 + \rho_0 A)) \\ &\quad - \text{Tr}(A\rho_0)\text{Tr}(\mathcal{G}(-\tau)(A\rho_0 + \rho_0 A)). \end{aligned}$$

The $-\tau$ in the last two lines appears since the Fourier transform of the conjugate complex of a function is the time-reversed original function. In the next step we use the property $\mathcal{G}(\tau) = 0$ for negative τ and can more com-

pactly write

$$\begin{aligned} 2G(\tau) &= \text{Tr}(A\mathcal{G}(|\tau|)(A\rho_0 + \rho_0 A)) \\ &\quad - \text{Tr}(A\rho_0)\text{Tr}(\mathcal{G}(|\tau|)(A\rho_0 + \rho_0 A)). \end{aligned} \tag{26}$$

We find $\text{Tr}(A\mathcal{G}(|\tau|)(A\rho_0 + \rho_0 A)) = \text{Tr}(A(|\tau|)(A\rho_0 + \rho_0 A))$ after invoking the Heisenberg picture. The second factor of the second line can be written as $\text{Tr}(\mathcal{G}(|\tau|)(A\rho_0 + \rho_0 A)) = \text{Tr}(A\rho_0 + \rho_0 A)$ as \mathcal{G} strictly conserves the trace. We moreover invoke the relation $\text{Tr}(A\rho_0) = \text{Tr}(A\mathcal{G}(\tau)\rho_0)$ which is possible as ρ_0 is the equilibrium state which does not change when propagated by $\mathcal{G}(\tau)$. We then can write $\text{Tr}(A\rho_0) = \text{Tr}(A(t)\rho_0)$ in the Heisenberg picture. All the above leads us to

$$G(\tau) = \frac{1}{2} \langle A(|\tau|)A(0) + A(0)A(|\tau|) \rangle - \langle A(|\tau|) \rangle \langle A(0) \rangle \tag{27}$$

in the Heisenberg picture.